

CERN-TH/96-63
 UWThPh-21-1996
 hep-th/9603071

$N = 2$ Superalgebra and Non-Commutative Geometry

H. Grosse¹

*Institute for Theoretical Physics, University of Vienna,
 Boltzmannngasse 5, A-1090 Vienna, Austria,*

C. Klimčík

*Theory Division CERN,
 CH-1211 Geneva 23, Switzerland*

and

P. Prešnajder

*Dept. of Theoretical Physics, Comenius University,
 Mlynská dolina F1, SK-84215 Bratislava, Slovakia*

Abstract

A construction of supersymmetric field-theoretical models in non-commutative geometry is reviewed. The underlying superstructure of the models is encoded in $osp(2, 2)$ superalgebra.

*Based on a seminar presented by C. Klimčík at LXIV. Les Houches session
 on ‘Quantum Symmetries’, August 1995*

CERN-TH/96-63
 March 1996

¹Participating in Project No. P8916-PHY of the ‘Fonds zur Förderung der wissenschaftlichen Forschung in Österreich’.

1 Introduction

Probably the most fruitful method for obtaining concrete nonperturbative quantitative predictions for more or less realistic quantum field theories consists in using some appropriate *finite dimensional* regularization of the theory and then performing the path integral quantization. As an example of such an approach we may mention the usual lattice regularization (see [1] for a review) which regularizes the space itself and hence the algebra of functions on it. The power of the lattice methods does not suffice to treat the supersymmetric theories however, because the notion of the superspace itself does not make a direct sense. Only the algebra of functions on the superspace is well defined.

In this contribution, we wish to describe an alternative of the usual lattice regularization which would be well suited also for regularizing the supersymmetric theories. Our basic idea relies in regularizing directly the superalgebra of functions on the superspace rather than considering a regularization of the superspace itself. Moreover, we shall truncate the superalgebra of functions in a manner consistent with supersymmetry and describe an appropriate modified differential and integral calculus by using the methods of the non-commutative geometry [2], i.e. the generalization of the ordinary differential geometry to non-commutative rings of ‘functions’. This calculus suits well for our basic purpose of writing down regularized actions of supersymmetric field theories.

As the starting point of our treatment we choose a 2d field theory on a truncated two-sphere²[6, 7]. Apparently, the truncated sphere was introduced by Berezin, in 1975 [3], who quantized the (symplectic) volume two-form on the ordinary two-sphere. The concept was rediscovered by Madore [5] (see also [8]) who has approached the structure from the point of view of the non-commutative geometry.

In what follows, we shall construct the supersymmetric extension of the truncated sphere and build up a field theory on it [9]. The basic underlying (super)structure is the Lie superalgebra $osp(2, 2)$. The resulting theory is manifestly finite and $osp(2, 1)$ supersymmetric.

²Also referred to as “fuzzy”, “non-commutative” or “quantum” sphere in the literature [3, 4, 5].

2 Commutative supersphere

Consider a three-dimensional superspace \mathbf{SR}^3 with coordinates x^i, θ^α ; the supercoordinates are the $su(2)$ Majorana spinors. Denote \mathcal{SB} the algebra of analytic functions on the superspace with the Grassmann coefficients in front of the odd monomials in θ .

The vector fields in \mathbf{SR}^3 generating $osp(2, 2)$ super-rotations of \mathcal{SB} are given by the explicit formulae [9]

$$v_+ = -\frac{1}{2}(x^3\partial_{\theta^-} - (x^1 + ix^2)\partial_{\theta^+}) + \frac{1}{2}(-\theta^+\partial_{x^3} - \theta^-(\partial_{x^1} + i\partial_{x^2})), \quad (1)$$

$$v_- = -\frac{1}{2}(x^3\partial_{\theta^+} + (x^1 - ix^2)\partial_{\theta^-}) + \frac{1}{2}(\theta^-\partial_{x^3} - \theta^+(\partial_{x^1} - i\partial_{x^2})), \quad (2)$$

$$d_+ = -\frac{1}{2}r(1 + \frac{2}{r^2}\theta^+\theta^-)\partial_- + \frac{\theta^-}{2r}R_+ - \frac{\theta^+}{2r}(x^i\partial_i - R_3), \quad (3)$$

$$d_- = \frac{1}{2}r(1 + \frac{2}{r^2}\theta^+\theta^-)\partial_+ + \frac{\theta^+}{2r}R_- - \frac{\theta^-}{2r}(x^i\partial_i + R_3) \quad (4)$$

$$\Gamma_\infty = \left(\frac{\theta^+x^3}{r} + \frac{\theta^-x^+}{r}\right)\partial_+ + \left(\frac{\theta^+x^-}{r} - \frac{\theta^-x^3}{r}\right)\partial_- \equiv 2(\theta^-v_+ - \theta^+v_-). \quad (5)$$

$$r_+ = x^3(\partial_{x^1} + i\partial_{x^2}) - (x^1 + ix^2)\partial_{x^3} + \theta^+\partial_{\theta^-}, \quad (6)$$

$$r_- = -x^3(\partial_{x^1} - i\partial_{x^2}) + (x^1 - ix^2)\partial_{x^3} + \theta^-\partial_{\theta^+}, \quad (7)$$

$$r_3 = -ix^1\partial_{x^2} + ix^2\partial_{x^1} + \frac{1}{2}(\theta^+\partial_{\theta^+} - \theta^-\partial_{\theta^-}) \quad (8)$$

and they obey the $osp(2, 2)$ Lie superalgebra graded commutation relations [10, 11]

$$[r_3, r_\pm] = \pm r_\pm, \quad [r_+, r_-] = 2r_3, \quad (9)$$

$$[r_3, v_\pm] = \pm \frac{1}{2}v_\pm, \quad [r_\pm, v_\pm] = 0, \quad [r_\pm, v_\mp] = v_\pm, \quad (10)$$

$$\{v_\pm, v_\pm\} = \pm \frac{1}{2}r_\pm, \quad \{v_\pm, v_\mp\} = -\frac{1}{2}r_3. \quad (11)$$

$$[\Gamma_\infty, v_\pm] = d_\pm, \quad [\Gamma_\infty, d_\pm] = v_\pm, \quad [\Gamma_\infty, r_i] = 0, \quad (12)$$

$$[r_3, d_\pm] = \pm \frac{1}{2}d_\pm, \quad [r_\pm, d_\pm] = 0, \quad [r_\pm, d_\mp] = d_\pm, \quad (13)$$

$$\{d_{\pm}, v_{\pm}\} = 0, \quad \{d_{\pm}, v_{\mp}\} = \pm \frac{1}{4} \Gamma_{\infty}, \quad (14)$$

$$\{d_{\pm}, d_{\pm}\} = \mp \frac{1}{2} r_{\pm}, \quad \{d_{\pm}, d_{\mp}\} = \frac{1}{2} r_3. \quad (15)$$

Note that the superfunction

$$\sum x^{i^2} + C_{\alpha\beta} \theta^{\alpha} \theta^{\beta} - \rho^2, \quad C = i\sigma^2 \quad (16)$$

is $osp(2, 2)$ -invariant. This means that the algebra \mathcal{SB} , factorized by its ideal \mathcal{SI} consisting of all functions of the form $h(x^i, \theta^{\alpha})(\sum x^{i^2} + C_{\alpha\beta} \theta^{\alpha} \theta^{\beta} - \rho^2)$, inherits the $osp(2, 2)$ action³. We denote the quotient as \mathcal{SA}_{∞} and refer to it as to the algebra of superfields on the supersphere.

An $osp(2, 2)$ invariant inner product of two elements Φ_1, Φ_2 of \mathcal{SA}_{∞} is given by⁴

$$(\Phi_1, \Phi_2)_{\infty} \equiv \frac{\rho}{2\pi} \int_{R^3} d^3 x^i d\theta^+ d\theta^- \delta(x^{i^2} + C_{\alpha\beta} \theta^{\alpha} \theta^{\beta} - \rho^2) \Phi_1^{\dagger}(x^i, \theta^{\alpha}) \Phi_2(x^i, \theta^{\alpha}), \quad (17)$$

Here $\Phi_1(x^i, \theta^{\alpha}), \Phi_2(x^i, \theta^{\alpha}) \in \mathcal{SB}$ are some representatives of Φ_1 and Φ_2 and the (graded) involution [12, 11] is defined by

$$\theta^{+\dagger} = \theta^-, \quad \theta^{-\dagger} = -\theta^+, \quad (AB)^{\dagger} = (-1)^{deg A \ deg B} B^{\dagger} A^{\dagger}. \quad (18)$$

The algebra \mathcal{SA}_{∞} is obviously generated by (the equivalence classes) x^i ($i = 1, 2, 3$) and θ^{α} ($\alpha = +, -$) which (anti)commute with each other under the usual pointwise multiplication, i.e.

$$x^i x^j - x^j x^i = x^i \theta^{\alpha} - \theta^{\alpha} x^i = \theta^{\alpha} \theta^{\beta} + \theta^{\beta} \theta^{\alpha} = 0. \quad (19)$$

Their norms are given by

$$||x^i||_{\infty}^2 = ||\theta^{\alpha}||_{\infty}^2 = \rho^2. \quad (20)$$

³ The appearance of r in Eqs. (3)-(5) may seem awful because we have considered the ring of superanalytic functions on SR_3 . However, r becomes harmless after the factorization by the ideal SI .

⁴The normalization ensures that the norm of the unit element of \mathcal{SA}_{∞} is 1. The inner product is supersymmetric but it is not positive definite. However, such a property of the product is not needed for our purposes.

As is well known [11], the typical irreducible representations of $osp(2, 2)$ consist of quadruplets of the $su(2)$ irreducible representations $j \oplus j - \frac{1}{2} \oplus j - \frac{1}{2} \oplus j - 1$. The number j is an integer or a half-integer, and it is referred to as the $osp(2, 2)$ superspin. The supermultiplet with the superspin 1 can be conveniently constructed, applying subsequently the lowering operators v_- and d_- on the highest weight vector x^+ . Supermultiplets with higher superspins can be obtained in the same way, starting with the highest weight vectors x^{+l} . Thus the full decomposition of \mathcal{SA}_∞ into the irreducible representations of $osp(2, 2)$ can be written as the infinite direct sum

$$\mathcal{SA}_\infty = 0 + 1 + 2 + \dots, \quad (21)$$

where the integers denote the $osp(2, 2)$ superspins of the representations. From the point of view of the $su(2)$ representations, the algebra of the superfields consists of two copies of the standard algebra of scalar fields on the ordinary sphere S^2 and one copy of the spinor bundle on S^2 . Note that the generators of \mathcal{SA}_∞ fulfil the obvious relation

$$x^{i2} + C_{\alpha\beta}\theta^\alpha\theta^\beta = \rho^2. \quad (22)$$

Now we turn to the problem of formulating supersymmetric field theories on the sphere. We restrict our attention to the case of $N = 1$ supersymmetry, hence we look for an $osp(2, 1)$ -invariant action for the superscalar field from \mathcal{SA}_∞ . It is easy to check that the operator

$$C_{\alpha\beta}d_\alpha d_\beta + \frac{1}{4}\Gamma_\infty^2 \quad (23)$$

is invariant with respect to $osp(2, 1)$ supersymmetry generated by r_i and v_\pm . We observe that the additional odd $osp(2, 2)$ generators d_α play simply the role of the supersymmetric covariant derivative in the following $osp(2, 1)$ invariant action

$$\begin{aligned} S &= (\Phi, C_{\alpha\beta}d_\alpha d_\beta \Phi)_\infty + \frac{1}{4}(\Phi, \Gamma_\infty^2 \Phi)_\infty \equiv \\ &\equiv \frac{\rho}{2\pi} \int_{R^3} d^3x^i d\theta^+ d\theta^- \delta(x^{i2} + C_{\alpha\beta}\theta^\alpha\theta^\beta - \rho^2) \Phi(x^i, \theta^\alpha) (C_{\alpha\beta}d_\alpha d_\beta + \frac{1}{4}\Gamma_\infty^2) \Phi(x^i, \theta^\alpha), \end{aligned}$$

where Φ is a real superfield, i.e. $\Phi^\dagger = \Phi$. Note that the additional bosonic $osp(2, 2)$ generator Γ_∞ also appears in the action.

Consider the variation of the real superfield Φ

$$\delta\Phi = i\varepsilon_\alpha v_\alpha \Phi, \quad (24)$$

which preserves the reality condition. Using the fact that $\varepsilon_\alpha v_\alpha$ commutes with the operator (23), the supersymmetry of the action S easily follows.

Now we should discuss the crucial importance of Γ_∞ in our formalism. Indeed, after the obvious identification of the θ -linear part of the superfield

$$\Phi(x^i, \theta^\alpha) = \phi(x^i) + \psi_\alpha \theta^\alpha + (F + \frac{x^i}{r^2} \partial_i \phi) \theta^+ \theta^- \quad (25)$$

with spinors on the sphere we observe that Γ_∞ is simply the grading operator of the spinor bundle. It anticommutes with the operator (23) restricted to fermions or, in other words, with the standard round Dirac operator on the sphere. Indeed, it is straightforward to work out the action in the two-dimensional component language. It reads

$$S = \frac{1}{4\pi} \int d\Omega \left(-\frac{1}{2} \phi \triangle_\Omega \phi + \frac{1}{2} \rho^4 F^2 - \frac{1}{2} \psi^\dagger \rho^3 D_\Omega \psi \right), \quad (26)$$

where D_Ω is the round Dirac operator on S^2 . We recognize in this expression the standard free supersymmetric action in two dimensions.

The fact that the chirality operator Γ_∞ is at the same time the $osp(2, 2)$ -generator is very important because any $osp(2, 2)$ -covariant regularization of the SUSY field theories will automatically be also chiral. Needless to say, particularly this aspect appears to be very promising from the point of view of higher dimensional generalizations of our formalism.

We conclude this section by noting that by adding a (real) superpotential $W(\Phi)$ we may write down a supersymmetric action with the interaction term. It reads

$$S_\infty = \left(\Phi, (C_{\alpha\beta} d_\alpha d_\beta + \frac{1}{4} \Gamma_\infty^2) \Phi \right)_\infty + (1, W(\Phi))_\infty. \quad (27)$$

3 Non-commutative supersphere

We define the non-commutative supersphere \mathcal{SA}_j by means of the truncation of the expansion (21). This means that the $osp(2, 2)$ decomposition of the superalgebra of superfunctions terminates at the value of the maximal

$osp(2, 2)$ superspin j which will play the role of cut-off in our regularization. Hence

$$\mathcal{SA}_j = 0 + 1 + \dots + j, \quad j \in \mathbf{Z} \quad (28)$$

as the vector space. We have to furnish this linear space with an associative product and an inner product, which in the limit $j \rightarrow \infty$, give the standard products in \mathcal{SA}_∞ .

In order to do this consider the space $\mathcal{L}(j/2, j/2)$ of linear operators from the representation space of the $osp(2, 1)$ irreducible representation with the $osp(2, 1)$ superspin $j/2$ into itself. (Note that the $osp(2, 1)$ irreducible representation with the $osp(2, 1)$ superspin j has the $su(2)$ content $j \oplus j - \frac{1}{2}$ [11].) The action of the superalgebra $osp(2, 2)$ itself on the $j/2$ representation space⁵ is described by the operators $R_i, V_\alpha, D_\alpha, \gamma \in \mathcal{L}(j/2, j/2)$ given by [14]:

$$R_i = \begin{pmatrix} R_i^{\frac{j}{2}} & 0 \\ 0 & R_i^{\frac{j}{2}-\frac{1}{2}} \end{pmatrix}, \quad \gamma = \begin{pmatrix} -j Id & 0 \\ 0 & -(j+1)Id \end{pmatrix}. \quad (29)$$

$$V_\alpha = \begin{pmatrix} 0 & V_\alpha^{\frac{j}{2}, \frac{j}{2}-\frac{1}{2}} \\ V_\alpha^{\frac{j}{2}-\frac{1}{2}, \frac{j}{2}} & 0 \end{pmatrix}, \quad D_\alpha = \begin{pmatrix} 0 & V_\alpha^{\frac{j}{2}, \frac{j}{2}-\frac{1}{2}} \\ -V_\alpha^{\frac{j}{2}-\frac{1}{2}, \frac{j}{2}} & 0 \end{pmatrix}, \quad (30)$$

where

$$\langle l, l_3 + 1 | R_+^l | l, l_3 \rangle = \sqrt{(l - l_3)(l + l_3 + 1)}, \quad (31)$$

$$\langle l, l_3 - 1 | R_-^l | l, l_3 \rangle = \sqrt{(l + l_3)(l - l_3 + 1)}, \quad (32)$$

$$\langle l, l_3 | R_3^l | l, l_3 \rangle = l_3, \quad (33)$$

$$\langle l_3 + \frac{1}{2} | V_+^{\frac{j}{2}, \frac{j}{2}-\frac{1}{2}} | l_3 \rangle = -\frac{1}{2} \sqrt{\frac{j}{2} + l_3 + \frac{1}{2}}, \quad (34)$$

$$\langle l_3 - \frac{1}{2} | V_-^{\frac{j}{2}, \frac{j}{2}-\frac{1}{2}} | l_3 \rangle = -\frac{1}{2} \sqrt{\frac{j}{2} - l_3 + \frac{1}{2}}, \quad (35)$$

$$\langle l_3 + \frac{1}{2} | V_+^{\frac{j}{2}-\frac{1}{2}, \frac{j}{2}} | l_3 \rangle = -\frac{1}{2} \sqrt{\frac{j}{2} - l_3}, \quad (36)$$

$$\langle l_3 - \frac{1}{2} | V_-^{\frac{j}{2}-\frac{1}{2}, \frac{j}{2}} | l_3 \rangle = \frac{1}{2} \sqrt{\frac{j}{2} + l_3}. \quad (37)$$

⁵The so-called non-typical irreducible representation of $osp(2, 2)$ [11, 13] is at the same time the $osp(2, 1)$ irreducible representation with the $osp(2, 1)$ superspin $j/2$.

Every $\Phi \in \mathcal{L}(j/2, j/2)$ can be written as a matrix

$$\Phi = \begin{pmatrix} \phi_R & \psi_R \\ \psi_L & \phi_L \end{pmatrix}, \quad (38)$$

where ϕ_R and ϕ_L are square $(j+1) \times (j+1)$ and $j \times j$ matrices, respectively, and ψ_R and ψ_L are respectively rectangular $(j+1) \times j$ and $j \times (j+1)$ matrices. A fermionic element is given by a supermatrix with vanishing diagonal blocks, and a bosonic element by one with vanishing off-diagonal blocks. Clearly, $osp(2, 2)$ superalgebra acts on $\mathcal{L}(j/2, j/2)$ by the superadjoint action

$$\mathcal{R}_i \Phi \equiv [R_i, \Phi], \quad \Gamma \Phi \equiv [\gamma, \Phi]. \quad (39)$$

$$\mathcal{V}_\alpha \Phi_{even} \equiv [V_\alpha, \Phi_{even}], \quad \mathcal{V}_\alpha \Phi_{odd} \equiv \{V_\alpha, \Phi_{odd}\}. \quad (40)$$

$$\mathcal{D}_\alpha \Phi_{even} \equiv [D_\alpha, \Phi_{even}], \quad \mathcal{D}_\alpha \Phi_{odd} \equiv \{D_\alpha, \Phi_{odd}\}. \quad (41)$$

This ‘superadjoint’ representation is reducible and, in the spirit of Refs.[11, 13], it is easy to work out its decomposition into $osp(2, 2)$ irreducible representations

$$\mathcal{L}(j/2, j/2) = 0 + 1 + \dots + j = \mathcal{SA}_j. \quad (42)$$

The associative product in $\mathcal{L}(j/2, j/2)$ is defined as the composition of operators, and the $osp(2, 2)$ invariant inner product on $\mathcal{L}(j/2, j/2)$ is defined by

$$(\Phi_1, \Phi_2)_j \equiv \text{STr}(\Phi_1^\dagger, \Phi_2), \quad \Phi_1, \Phi_2 \in \mathcal{L}(j/2, j/2). \quad (43)$$

The supertrace STr is defined as usual

$$\text{STr} \Phi \equiv \text{Tr} \phi_R - \text{Tr} \phi_L \quad (44)$$

and the graded involution \dagger as [12]

$$\Phi^\dagger \equiv \begin{pmatrix} \phi_R^\dagger & \mp \psi_L^\dagger \\ \pm \psi_R^\dagger & \phi_L^\dagger \end{pmatrix}. \quad (45)$$

\dagger means the standard Hermitian conjugation of a matrix and the upper (lower) sign refers to the case when the entries consists of odd (even) elements of a Grassmann algebra. Note that

$$R_i^\dagger = R_i, \quad V_+^\dagger = V_-, \quad V_-^\dagger = -V_+. \quad (46)$$

Now we identify \mathcal{SA}_j with even elements of $\mathcal{L}(j/2, j/2)$, which means that the entries of the diagonal matrices are commuting numbers and those of the off-diagonal matrices in turn are Grassmann variables.

The $osp(2, 1)$ Casimir reads

$$R_i^2 + C_{\alpha\beta} V_\alpha V_\beta \quad (47)$$

and its value in the $j/2$ representation is $j(2j + 1)/4$. We may renormalize the $osp(2, 1)$ generators R_i and V_α essentially by the square-root of the value of the Casimir; the renormalized generators we denote as X_j^i and Θ_j^α and j refers to the superspin. More precisely, we require that in each irreducible representation the new generators fulfil the defining equation of the supersphere:

$$X_j^{i^2} + C_{\alpha\beta} \Theta_j^\alpha \Theta_j^\beta = \rho^2. \quad (48)$$

It can be shown [9] that the objects X_∞^i and Θ_∞^α gives the standard commutative generators x^i and θ^α of the previous section. The simplest test of consistence of this statement may be provided by looking at the commutation relations of the renormalized generators:

$$[X_j^m, X_j^n] = i \frac{\rho}{\sqrt{\frac{j}{2}(\frac{j}{2} + \frac{1}{2})}} \epsilon_{mnp} X_j^p, \quad (49)$$

$$[X_j^i, \Theta_j^\alpha] = \frac{\rho}{2\sqrt{\frac{j}{2}(\frac{j}{2} + \frac{1}{2})}} \sigma^{i\beta\alpha} \Theta_j^\beta, \quad (50)$$

$$\{\Theta_j^\alpha, \Theta_j^\beta\} = \frac{\rho}{2\sqrt{\frac{j}{2}(\frac{j}{2} + \frac{1}{2})}} (C\sigma^i)^{\alpha\beta} X_j^i. \quad (51)$$

Consider now the variation of a superfield Φ

$$\delta\Phi = i(\epsilon_+ \mathcal{V}_+ + \epsilon_- \mathcal{V}_-) \Phi, \quad (52)$$

where ϵ_α is given by

$$\epsilon_\alpha = \begin{pmatrix} \varepsilon_\alpha & 0 \\ 0 & -\varepsilon_\alpha \end{pmatrix} \quad (53)$$

and ε_α are the usual Grassmann variables with the involution properties

$$\varepsilon_+^\dagger = \varepsilon_-, \quad \varepsilon_-^\dagger = -\varepsilon_+. \quad (54)$$

Because the inner product (43) is $osp(2, 2)$ invariant, we can easily demonstrate the invariance of the following action (defined on the truncated supersphere) under the $osp(2, 1)$ variation (52):

$$S_j = \left(\Phi, (C_{\alpha\beta} \mathcal{D}_\alpha \mathcal{D}_\beta + \frac{1}{4} \Gamma^2) \Phi \right)_j + (1, W(\Phi))_j. \quad (55)$$

The inner product appearing here was defined in (43), the operators \mathcal{D}_α and Γ in (39) - (41) and the field Φ in (38). We may also write down the regularized action for the supersymmetric σ -models describing the superstring propagation in curved backgrounds:

$$S_j = (\mathcal{D}_+ \Phi^A, g_{AB}(\Phi) \mathcal{D}_+ \Phi^B)_j + (\mathcal{D}_- \Phi^A, g_{AB}(\Phi) \mathcal{D}_- \Phi^B)_j + \frac{1}{4} (\Gamma \Phi^A, g_{AB}(\Phi) \Gamma \Phi^B)_j, \quad (56)$$

where $g_{AB}(\Phi)$ denotes the metric on the target space manifold in which superstring propagates. The $osp(2, 1)$ supersymmetry and the commutative limit are obvious.

The regularized actions (55) and (56) can be used as the base for the path integral quantization, manifestly preserving supersymmetry and still involving only the finite number of degrees of freedom. Particularly this aspect of our approach seems to be very promising both in comparison with lattice physics and also in general.

4 Outlook

There are at least two obvious ways how to proceed in order to make the method better elaborated and more universal. The first one would consist in introducing supergauge fields in the formalism and the second one in supersymmetrizing the truncated four-dimensional bosonic sphere [15]. We believe that a success of the four-dimensional program would result in immediate realistic physical applications.

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